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On Fractional Integral Inequalities Involving Riemann-Liouville Operators

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ABSTRACT: Here, we seek to prove some novel fractional integral inequalities for synchronous functions connected to the Chebyshev functional, involving the Gauss hypergeometric function. The final section presents a number of special instances as fractional integral inequalities involving Riemann-Liouville type fractional integral operators. Additionally, we take into account their applicability to other relevant, previous findings.

KEYWORDS: Fractional Integral Inequalities & Riemann-Liouville Operators

1. INTRODUCTION

The most beneficial uses of fractional integral inequalities are in determining the uniqueness of solutions to fractional boundary value issues and fractional partial differential equations. Additionally, they offer upper and lower bounds for the solutions of the aforementioned equations.

These factors have prompted a number of scholars working in the area of integral inequalities to investigate various extensions and generalisations by utilising fractional calculus operators. For instance, the book [1] and the publications [2-11] both contain references to such works.

Purohit and Raina [9] recently looked into some integral inequalities of the Chebyshev type [12] utilising Saigo fractional integral operators and established the q-extensions of the main findings. This study uses the fractional hypergeometric operator proposed by Curiel and Galue [13] to prove a few generalised integral inequalities for synchronous functions related to the Chebyshev functional. As special examples of our findings, the results attributed to Purohit and Raina [9] and Belarbi and Dahmani [2] are presented below.

DEFINITIONS AND PRELIMINARIES USED IN THIS PAPER

Definition 1.1 Two functions f & g are said to be synchronous on $[a,b]$ if

$$(f(x)-f(y))(g(x)-g(y)) \;\; [a,b] \qquad (1.1)$$

Definition 1.2 A real-valued function f(t) (t> 0) is said to be in the space C_μ ($\mu \in$ R) if there exists a real number $p > \mu$, such that

$$f(t) = t^p \emptyset(t), \text{ where } \emptyset(t) \in C(0,\infty).$$

Definition 1.3 Let $\alpha > 0$, $\mu >$-1, $\beta,\eta \in$ R; then a generalized fractional integral $I_t^{\alpha,\beta,\eta,\mu}$ (in terms of the Gauss hypergeometric) of order α for a real valued continuous function f(t) is defined in [13]

$$I_{0,x}^{\alpha,\beta,\eta,\mu} f(x) = \frac{x^{-\alpha-\beta-2\mu}}{\Gamma\alpha}\int_0^x \tau^\mu (x-\tau)^{\alpha-1} {}_2F_1\left[\alpha+\beta,-\eta\,;1-\frac{t}{x}\right] f(t)\,dt, \qquad (1.2)$$

2. MAIN RESULTS

In this section we obtain certain Chebyshev type integral inequalities involving the generalized fractional integral operator. The following lemma is used for our first result.

Lemma 2.1 let $\alpha > 0$, $\mu >$-1, $\beta,\eta \in$ R; then the following image formula for the power function under the operator(1.2) holds true:

$$I_{0,x}^{\alpha,\beta,\eta}\{x^{\tau-1}\} = \frac{\Gamma(\mu+\tau)\Gamma(\tau-\beta+\eta)}{\Gamma(\tau-\beta)\Gamma(\tau+\mu+\alpha+\eta)} x^{\tau-\beta-\mu-1} \qquad (2.1.1)$$

Theorem 2.1. Let f & g be two synchronous functions on $(0,\infty)$ then

$$I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)g(x)\} \geq \frac{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}{\Gamma(1+\mu)\Gamma(1-\beta+\eta)} x^{\beta+\mu} \times I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)\} I_{0,x}^{\alpha,\beta,\eta}\{g(x)\}$$

For all $x > 0$, $\alpha > \max\{0, -\beta-\mu\}$, $\beta < 1 \mu > -1, \eta < 0$.

Proof: Let f & g be two synchronous functions , then using definition1, for all $\tau, \rho \epsilon (0,t), t \geq 0$, we have

$$\{(f(\tau) - f(\rho)(g(\tau) - g(\rho)\} \geq 0,$$

which implies that

$$(f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau).$$

Consider

$$F(x,\tau) = \frac{x^{-\alpha-\beta-2\mu}\tau^{\mu}(x-t)^{\alpha-1}}{\Gamma\alpha} \; {}_2F_1\left[\alpha+\beta+\mu, -\eta \,; 1-\frac{\tau}{x}\right] f(t)dt =$$

$$\frac{\tau^{\mu}(x-\tau)^{\alpha-1}}{\Gamma\alpha\, x^{\alpha+\beta+2\mu}} \times \frac{\tau^{\mu}(\alpha+\beta+\mu)(-\eta)(x-\tau)^{\alpha}}{\Gamma(\alpha+1)\; x^{\alpha+\beta+2\mu}} + \frac{\tau^{\mu}(\alpha+\beta+\mu)(\alpha+\beta+\mu+1)(x-\tau)^{\alpha-1}}{\Gamma(\alpha+2)} \times \frac{(x-\tau)^{\alpha+1}}{x^{\alpha+\beta+2\mu+2}}$$

Our observation is that each term of the above series is positive in view of the conditions stated with Theorem 2.1

$$I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)g(x)\} \geq \frac{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}{\Gamma(1+\mu)\Gamma(1-\beta+\eta)} x^{\beta+\mu} \times I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)\} I_{0,x}^{\alpha,\beta,\eta}\{g(x)\}$$

in light of the circumstances outlined in Theorem 1, we note that each term in the aforementioned series is positive. On integrating from 0 to τ and using definition3, we get

$$I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)g(x)\} + f(\rho)g(\rho) I_{0,x}^{\alpha,\beta,\eta,\mu}\{I\} \geq g(\rho) I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)\} + f(\rho) I_{0,x}^{\alpha,\beta,\eta,\mu}\{g(x)\}$$

Theorem 2.2: let f & g be two synchronous functions on $(0,\infty)$ then

$$\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)x^{\beta+\mu}} I_{0,x}^{\delta,\gamma,\vartheta,\xi}\{f(x)g(x)\}$$

$$+ \quad \frac{\Gamma(1+\vartheta)\Gamma(1-\delta+\xi)}{\Gamma(1-\delta)\Gamma(1+\vartheta+\gamma+\xi)x^{\beta+\mu}} I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)g(x)\}$$

$$\geq I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)\} I_{0,x}^{\delta,\gamma,\vartheta,\xi}\{g(x)\} + I_{0,x}^{\delta,\gamma,\vartheta,\xi}\{g(x)\} I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)\}$$

For all $x > 0$, $\alpha > \max\{0, -\beta-\mu\}$, $\gamma > \mathrm{Max}(0, -\delta, -\vartheta)$, $\mu, \vartheta > -1$.

Proof: **T**o prove the above theorem, we use previous theorem , we have

$$\frac{x^{-\gamma-2\vartheta-\delta}\rho^{\vartheta}(x-\rho)^{\gamma-1}}{\Gamma\gamma} \; {}_2F_1\left[\delta+\delta+\vartheta, -\xi; 1-\frac{\rho}{x}\right] f(\rho)\, d\rho.$$

which, given the circumstances mentioned with, continues to be positive as per above theorem and we obtain by integrating with regard to from 0 to t.

$$I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)g(x)\} I_{0,x}^{\delta,\gamma,\vartheta,\xi}\{I\} + I_{0,x}^{\delta,\gamma,\vartheta,\xi}\{g(x)f(x)\} I_{0,x}^{\alpha,\beta,\eta,\mu}\{I\} \geq I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)\} I_{0,x}^{\delta,\gamma,\vartheta,\xi}\{g(x)\} + I_{0,x}^{\delta,\gamma,\vartheta,\xi}\{G(x)\} I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)\}.$$

SPECIAL CASES

Here, we take a quick look at a few implications of the findings in the preceding section. The operator (1.2) would instantly decrease to the thoroughly studied, Erdelyi-Kober, and Riemann-Liouville type fractional integral operators, respectively, according to Curiel and Galué [13], given by the relationships below (see also [14, 16])

Case I:

$$I_{0,x}^{\alpha,\beta,\eta} f(x) = I_{0,x}^{\alpha,\beta,\eta,0} f(x) =$$

$$\frac{x^{-\alpha-\beta}}{\Gamma\alpha} \int_0^x (x-\tau)^{\alpha-1} {}_2F_1\left[\alpha+\beta, -\eta \,; 1-\frac{\tau}{x}\right] f(\tau)\, d(\tau) \,,$$

Here we get , Erdelyi-Kober fractional operator.

$$= I_{0,x}^{\alpha,\eta} f(x) = I_{0,x}^{\alpha,0,\eta,0} f(x) = \frac{x^{-\alpha-\eta}}{\Gamma\alpha} \int_0^x (x-\tau)^{\alpha-1} \tau^{n} f(\tau)\, d(\tau)$$

Case II:

Here we get, Riemann-Liouville fractional operator

$$= R^{\alpha} f(x) = I_{x}^{\alpha,-\alpha,\eta,0} f(x) = \frac{1}{\Gamma\alpha} \int_0^x (x-\tau)^{\alpha-1} f(\tau)\, d(\tau).$$

Case III:

for $\gamma=\alpha$, $\delta = \beta$, $\xi=\eta$ & $\vartheta = \mu$ in theorem2 immediately reduces in to theorem1.

$$I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)g(x)\} I_{0,x}^{\delta,\gamma,\vartheta,\xi}\{I\} + I_{0,x}^{\delta,\gamma,\vartheta,\xi}\{g(x)f(x)\} I_{0,x}^{\alpha,\beta,\eta,\mu}\{I\} \geq I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)\} I_{0,x}^{\delta,\gamma,\vartheta,\xi}\{g(x)\} + I_{0,x}^{\delta,\gamma,\vartheta,\xi}\{G(x)\} I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)\}.$$

Putting $\gamma=\alpha$, $\delta=\beta$, $\xi=\eta$ & $\vartheta=\mu$, we get

$$I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)g(x)\}+f(\rho)g(\rho)I_{0,x}^{\alpha,\beta,\eta,\mu}\{I\}\geq g(\rho)I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)\}+f(\rho)I_{0,x}^{\alpha,\beta,\eta,\mu}\{g(x)\}$$

CONCLUSION

Since the beginning of differential calculus, fractional calculus (FC) has been used.
mathematical theory However, throughout the past 20 years, FC's application has emerged as a result of the progress in chaos, demonstrating minute connections to FC tenets. FC has recently enjoyed success in the fields of science and engineering.
several scientific sectors are paying more for research topics. now pay attention to the FC ideas. There has been some work done. in the theory of dynamic systems, even if the suggested models Such algorithms are still being developed at a young stage. This Several case studies on FC based models were provided in the text. controls that highlight the benefits of utilising FC theory in a variety of scientific and engineering disciplines. This piece examined a number of physical systems, in particular

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